Constraints on Non-Singular Cosmological Models with Quadratic Lagrangians

R. Colistete Jr.*

L.P.T.L., Université Paris VI

Tour 22-12, 4ème étage, Boîte 142, 4 place Jussieu, 75005 Paris - France

We consider the generalized set of theories of gravitation whose Lagrangians contain the term R^2 : $L = \sqrt{-g}(R + \beta R^2)$. Inserting the RW metric with an imposed non-singular and inflationary behaviour of the scale factor $a(t)$, and using a arbitrary perfect fluid, we study the properties of ρ and p in this context. By requiring the positivity of the energy density , as well as real and finite velocity of sound, we can obtain the range of values of β that ensure the inflationary behaviour and absence of singularity.

I. INTRODUCTION

There are many motivations to study quadratic-order curvature Lagrangians in cosmology: quantum corrections [1,2] and renormalization of divergences, [3–5] effective actions of superstring theory, [6] inflation without scalar fields, [7] etc.

The aim of the present work is to explore the consequences of adding quadratic-order curvature terms to the Einstein action. From the beginning, a specified non-singular and inflationary scale factor and the Robertson-Walker metric are imposed, and the unknown energy-momentum tensor is then analysed.

The only energy-momentum tensor $T_{\mu\nu}$ choice in this work will be an arbitrary perfect fluid described by the energy density ρ and the isotropic pressure p. Just a couple of simple energy conditions will be used here,

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*Email address: `coliste@ccr.jussieu.fr`

$$\rho > 0 \ , \qquad (\rho + p) > 0 \ , \tag{1}$$

and additionally the adiabatic speed of sound, $v_s^2 = (\partial p/\partial \rho)$, must obey

$$0 \leq v_s^2 \leq 1 \ . \tag{2}$$

For each value of the spatial curvature k, the form of ρ and p are obtained as functions of the scale factor $a(t)$ and other constants. The energy (1) and velocity of sound (2) conditions can give constraints on the constant β associated with the quadratic-order curvature terms. It is also possible to compare the violations of the conditions (1,2) with and without the quadratic-order curvature terms, which gives some insight on the consequences of using quadratic Lagrangians.

II. THE MODIFIED FRIEDMANN EQUATION

Considering the fact [1] that the quantity $\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$, known as Gauss-Bonnet term, is a total divergence in four-dimensional spacetime, the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term can be expressed in terms of R^2 and $R_{\mu\nu}R^{\mu\nu}$, so the Riemann tensor is not necessary to account for the curvature square corrections.

Representing a general quadratic order curvature Lagrangian (but without the cosmological constant), let us take the Lagrangian

$$L = \sqrt{-g}(R + \beta_1 R^2 + \beta_2 R_{\mu\nu}R^{\mu\nu}) \ . \tag{3}$$

Calculating the variation of this action with respect to the metric $g^{\mu\nu}$, the modified Einstein equations are

$$G_{\mu\nu} + \beta_1 W_{\mu\nu}^{(1)} + \beta_2 W_{\mu\nu}^{(2)} = \kappa T_{\mu\nu} \ . \tag{4}$$

We will use the Robertson-Walker (RW) metric

$$ds^2 = -dt^2 + a(t)^2 \left[\left(\frac{1}{1 - k \ r^2} \right) dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right] \ , \tag{5}$$

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where the spatial curvature k takes the values $0, +1, -1$ (corresponding to flat, closed and open spatial sections). In the case of this spherically symmetric homogeneous metric, the Weyl tensor has null components. So the identity $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ is equal to zero, yielding another equation that now allows us to write $R_{\mu\nu}R^{\mu\nu}$ in terms of R^2 :

$$L = \sqrt{-g}\left[R + \left(\beta_1 + \frac{\beta_2}{3}\right)R^2\right] = \sqrt{-g}(R + \beta R^2) \ . \tag{6}$$

With the use of a perfect fluid, the modified Friedmann third-order differential equation is obtained :

$$\frac{a'(t)^2 + k}{a(t)^2} - \frac{\kappa\,\rho(t)}{3} - \frac{18\,\beta}{a(t)^4}\Big\{k^2 - 3\,a'(t)^4 - a(t)^2\,a''(t)^2 - \tag{7}$$
$$2\,a'(t)^2\,[k - a(t)\,a''(t)] + 2\,a(t)^2\,a'(t)\,a'''(t)\Big\} = 0 \ ,$$

where $\beta = (\beta_1 + \frac{\beta_2}{3})$ different from zero means that the quadratic-order curvature corrections are present.

With the RW metric (5), the energy-momentum conservation equations yield

$$\rho'(a) = \frac{-3(\rho + p)}{a} \ , \tag{8}$$

so that $p(a)$ can be calculated using $\rho(a)$.

III. NON-SINGULAR SCALE FACTORS WITH INFLATIONARY BEHAVIOURS

Non-singular and inflationary scale factors for the three values of spatial curvature k are studied. They show inflationary behaviour matching the classical radiation dominated behaviour.

I) The $k = 0$ case :

The following equation [8]

$$t = \frac{a(t)^2}{a_0{}^2} + \alpha \, \log\left[\frac{a(t)^2}{a(t)^2 + a_0{}^2}\right] \quad , \tag{9}$$

allows to obtain a scale factor $a(t)$, which for $\alpha = 0$ (or $a \gg a_0$) yields $a(t) = a_0\sqrt{t}$ (the known solution for radiation dominated era in FRW cosmology with $k = 0$), and for $t \to -\infty$ (or $a \ll a_0$) gives $a(t) = a_0 \exp(t/2\alpha)$. Figure 1 shows this behaviour compared to $a(t) = a_0\sqrt{t}$.

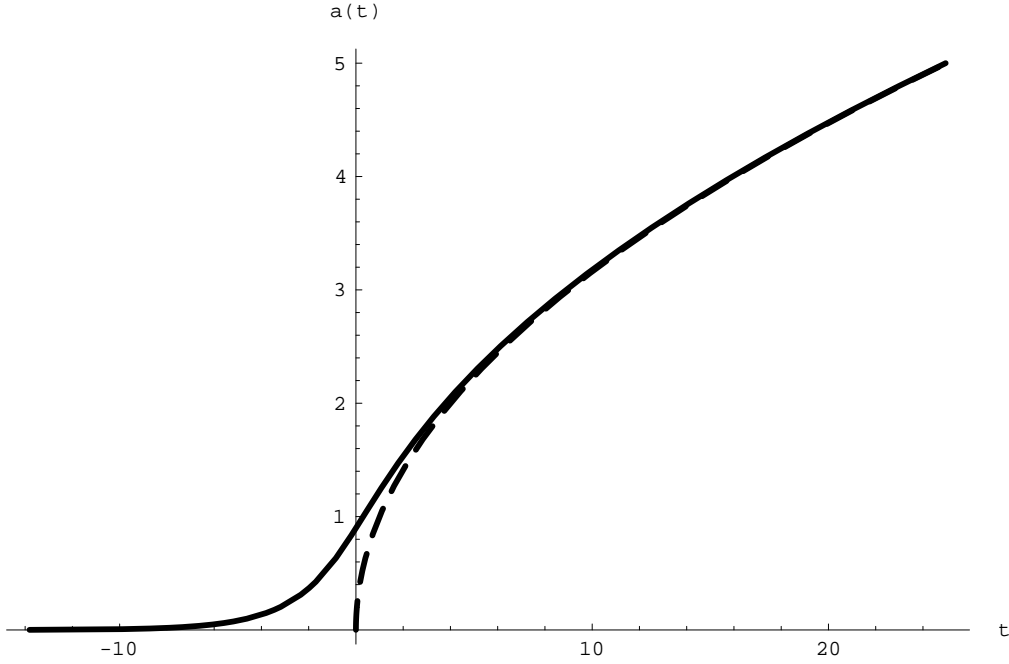FIG. 1. $a(t) \times t$ for the case $k = 0$, with $a_0 = \alpha = 1$. The solid line is the non-singular scale factor with inflationary behaviour; the dashed line is the known scale factor, $a(t) = a_0\sqrt{t}$, for radiation era in FRW cosmology.

It is possible to derive Eq. (9) in respect to t and write

$$a'(t) = \frac{a(t) \, a_0{}^2 \, [a(t)^2 + a_0{}^2]}{2 \, [a(t)^4 + a(t)^2 \, a_0{}^2 + a_0{}^4 \, \alpha]} \quad , \tag{10}$$

deriving this Eq. (10), the second derivative can be obtained,

$$a''(t) = \frac{a(t) \, a_0{}^8 [a(t)^2 + a_0{}^2][3 \, a(t)^2 + a_0{}^2] \, \alpha - a(t)^3 \, a_0{}^4 \, [a(t)^2 + a_0{}^2]^3}{4 \, [a(t)^4 + a(t)^2 \, a_0{}^2 + a_0{}^4 \, \alpha]^3} \quad , \tag{11}$$

and deriving this Eq. (11), the third derivative is given by

$$a'''(t) = \frac{a(t)\, a_0{}^6\, [a(t)^2 + a_0{}^2]}{8\, [a(t)^4 + a(t)^2\, a_0{}^2 + a_0{}^4\, \alpha]^5} \left\{ 3\, a(t)^4\, [a(t)^2 + a_0{}^2]^4 - \right.$$
$$2\, a(t)^2\, a_0{}^4\, [a(t)^2 + a_0{}^2][\, 15\, a(t)^4 + 15\, a(t)^2\, a_0{}^2 + 4\, a_0{}^4]\, \alpha +$$
$$\left. a_0{}^8\, [15\, a(t)^4 + 12\, a(t)^2\, a_0{}^2 + a_0{}^4]\, \alpha^2 \right\} \ . \tag{12}$$

The scale factor $a(t)$ and its derivatives are smooth functions of t.

II) The $k = +1$ case :

Generalizing the equation Eq. (9),

$$t = \left(1 - \sqrt{1 - \frac{a(t)^2}{a_0{}^2}} \right) t_0 + \alpha\, \log\left[\frac{a(t)^2}{a(t)^2 + a_0{}^2} \right] \ , \tag{13}$$

allows to obtain $a(t)$, which for $\alpha = 0$ (or $a \approx a_0$) yields $a(t) = a_0\sqrt{1 - (1 - t/t_0)^2}$ (the known solution for radiation dominated era in FRW cosmology with $k = +1$), and for $t \to -\infty$ (or $a << a_0$) gives $a(t) = a_0\exp(t/2\alpha)$.

As in the $k = 0$, it is possible to derive Eq. (13) in respect to t and write

$$a'(t) = \frac{a(t)a_0\sqrt{a_0{}^2 - a(t)^2}\, [a(t)^2 + a_0{}^2]}{a(t)^4 t_0 + a(t)^2\, a_0{}^2 t_0 + 2a_0{}^3\sqrt{a_0{}^2 - a(t)^2}\, \alpha} \ , \tag{14}$$

and the second and third derivatives can also be obtained without problems. All these functions are smooth.

III) The $k = -1$ case :

Also generalizing the equation Eq. (9),

$$t = \left(-1 + \sqrt{1 + \frac{a(t)^2}{a_0{}^2}} \right) t_0 + \alpha\, \log\left[\frac{a(t)^2}{a(t)^2 + a_0{}^2} \right] \ , \tag{15}$$

allows to obtain $a(t)$, which for $\alpha = 0$ (or $a >> a_0$) yields $a(t) = a_0\sqrt{-1 + (1 + t/t_0)^2}$ (the known solution for radiation dominated era in FRW cosmology with $k = -1$), and for $t \to -\infty$ (or $a << a_0$) gives $a(t) = a_0\exp(t/2\alpha)$.

As in the $k = 0$, it is possible to derive Eq. (15) in respect to t and write

$$a'(t) = \frac{a(t)a_0 \left[a(t)^2 + a_0{}^2\right]^{3/2}}{a(t)^4 t_0 + a(t)^2 a_0{}^2 t_0 + 2a_0{}^3 \sqrt{a_0{}^2 + a(t)^2} \, \alpha} \ , \tag{16}$$

and the second and third derivatives can also be easily obtained. All these derivatives are smooth functions.

IV. ANALYSIS OF ρ AND P

Instead of trying to obtain solutions for $a(t)$ of the modified Friedmann equation, Eq. (7), which can only be solved numerically, another way of analysis will be chosen. Replacing the derivatives of $a(t)$, then the energy density $\rho(t)$ can be written as $\rho(a)$, also depending on the constants k and β and the constants of the definition of $a\,(t)$.

I) The $k = 0$ case :

The energy density is given by

$$\rho(a) = \frac{3\,a_0{}^4 \left(a^2 + a_0{}^2\right)^2}{4\left(a^4 + a^2\,a_0{}^2 + a_0{}^4\,\alpha\right)^2 \kappa} + \frac{27\,a^2\,a_0{}^{12} \left(a^2 + a_0{}^2\right)^2 \alpha\,\beta}{2\left(a^4 + a^2\,a_0{}^2 + a_0{}^4\,\alpha\right)^6 \kappa} \times \tag{17}$$
$$\left[14\,a^6 + 29\,a^4\,a_0{}^2 + 2\,a^2\,a_0{}^4 \left(10 - 3\,\alpha\right) - 5\,a_0{}^6 \left(-1 + \alpha\right)\right] \ ,$$

and a thorough analysis shows that $\rho \geq 0$ if α and β are inside the region

$$Q_1 = \{0 \leq \alpha < 1, \beta > \beta_-\} \cup \{\alpha > 1, \beta_- < \beta < \beta_+\} \ , \tag{18}$$

where β_- and β_+ are functions of α. The fact of adding the non-linear terms in the Lagrangian is represented by $\beta \neq 0$, and does not force ρ to be negative.

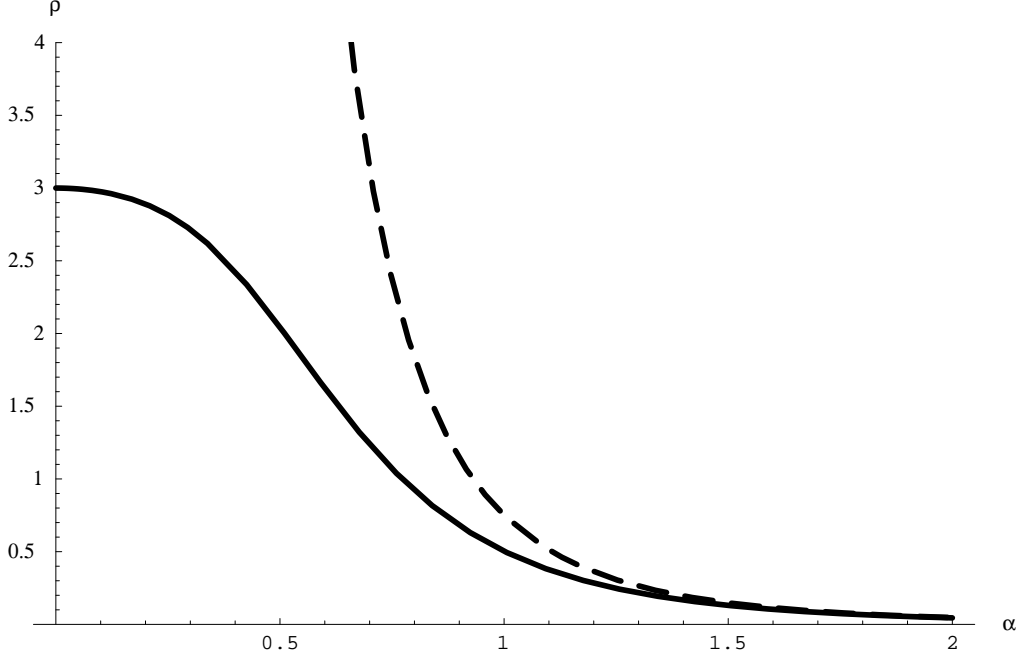The Figure 2 shows the behaviour of $\rho(a)$, which has no divergences if $\alpha > 0$.

FIG. 2. $\rho(a) \times a$ for the case $k = 0$, with $a_0 = \kappa = 1$, $\alpha = 0.5$, $\beta = 0.01$. The solid line uses the non-singular scale factor with inflationary behaviour; the dashed line is obtained using the known scale factor, $a(t) = a_0 \sqrt{t}$, for radiation era in FRW cosmology.

The pressure p can be obtained from the Eq. (8),

$$p(a) = \frac{a_0{}^4 \, (a^2 + a_0{}^2) \, [a^6 + 2\,a^4\,a_0{}^2 + a^2\,a_0{}^4 \, (1 - 7\,\alpha) - 3\,a_0{}^6\,\alpha]}{4\,(a^4 + a^2\,a_0{}^2 + a_0{}^4\,\alpha)^3 \, \kappa} + \tag{19}$$
$$\frac{9\,a^2\,a_0{}^{12} \, (a^2 + a_0{}^2) \, \alpha \, \beta}{2\,(a^4 + a^2\,a_0{}^2 + a_0{}^4\,\alpha)^7 \, \kappa} \Big\{ 126\,a^{12} + 459\,a^{10}\,a_0{}^2 + 8\,a^8\,a_0{}^4 \, (85 - 36\,\alpha) +$$
$$25\,a_0{}^{12} \, (-1 + \alpha) \, \alpha - 6\,a^6\,a_0{}^6 \, (-87 + 119\,\alpha) +$$
$$3\,a^4\,a_0{}^8 \, [70 + \alpha \, (-207 + 22\,\alpha)] + a^2\,a_0{}^{10} \, [35 + \alpha \, (-220 + 87\,\alpha)] \Big\} \,,$$

If $\alpha > 0$, the pressure p is always negative for some region in the $a(t)$ or t domain, independent of β. For $\alpha > 0$, p has no divergences.

Testing whether $\rho + p \geq 0$ is true just restricts α and β inside the region

$$Q_2 = \{0 \leq \alpha < 1, \beta_- < \beta < \beta_+\} \,, \tag{20}$$

where β_- and β_+ are another functions of α, different from the analysis of ρ.

The velocity of sound,

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$$v_s^2 = \frac{\partial p}{\partial \rho} = \frac{\partial p/\partial a}{\partial \rho/\partial a} \,, \tag{21}$$

does not diverge if α and β are inside the same region Q_2. But it can be imaginary for any value of $(\alpha, \beta) \neq (0,0)$.

So the conclusion for the $k = 0$ case is that the addition of high-order terms in the Einstein action does not lead to any other anomaly in the effective perfect fluid, i.e., with or without β, the behaviour of ρ, p, $\rho + p$ and v_s is the same (under the condition that α and β are inside the region Q_2).

II) The $k = +1$ case :

Analogous to the $k = 0$ case, but with more difficult calculations, the energy density is given by

$$\rho(a) = \frac{3\,a_0{}^2}{\left(a^5 + a^3\,a_0{}^2 + 2\,a\,a_0{}^2\,\sqrt{-a^2 + a_0{}^2}\,\alpha\right)^2 \kappa} \left[a^6 + 2\,a^4\,a_0{}^2 + a^2\,a_0{}^4 + \right. \tag{22}$$
$$\left. 4\,a^2\,\sqrt{-a^2 + a_0{}^2}\,\left(a^2 + a_0{}^2\right)\,\alpha - 4\,a^2\,a_0{}^2\,\alpha^2 + 4\,a_0{}^4\,\alpha^2\right] +$$
$$\frac{216\,a_0{}^2\,\alpha\,\beta}{a^4\left(a^4 + a^2\,a_0{}^2 + 2\,a_0{}^2\,\sqrt{-a^2 + a_0{}^2}\,\alpha\right)^6 \kappa} \left\{a^6\,(a - a_0)\,a_0{}^2\,(a + a_0) \times \right.$$
$$\left(a^2 + a_0{}^2\right)^2 \left(51\,a^6 + 14\,a^4\,a_0{}^2 - 13\,a^2\,a_0{}^4 + 8\,a_0{}^6\right)\,\alpha + 4\,a_0{}^6\left(a^5 - a\,a_0{}^4\right)^2 \times$$
$$\left(-17\,a^2 + 2\,a_0{}^2\right)\,\alpha^3 - 16\,a_0{}^{10}\left(-a^2 + a_0{}^2\right)^3\,\alpha^5 + a^2\,\sqrt{-a^2 + a_0{}^2} \times$$
$$\left(a^4 - a_0{}^4\right)\left[a^4\left(a^2 + a_0{}^2\right)^2\left(5\,a^6 - 34\,a^4\,a_0{}^2 - 33\,a^2\,a_0{}^4 - 10\,a_0{}^6\right) + \right.$$
$$\left.\left. 8\,a^2\,a_0{}^4\left(7\,a^2 - 2\,a_0{}^2\right)\left(a^2 + a_0{}^2\right)^2\,\alpha^2 + 48\,a_0{}^8\left(-a^2 + a_0{}^2\right)\,\alpha^4\right]\right\} \,,$$

and a long analysis shows that $\rho \geq 0$ if α and β are inside the region

$$Q_1 = \{0 \leq \alpha < \alpha_{\max}, \beta_- < \beta \leq 0\} \cup \{\alpha > \alpha_{\max}, \beta < 0\} \,, \tag{23}$$

where β_- is function of α and a_0, and $\alpha_{\max}$ is a function of a_0, $\alpha_{\max} \approx 0.73\ a0$. Figure 3 shows the region Q_1. The non-linear terms in the Lagrangian do not force ρ to be negative.
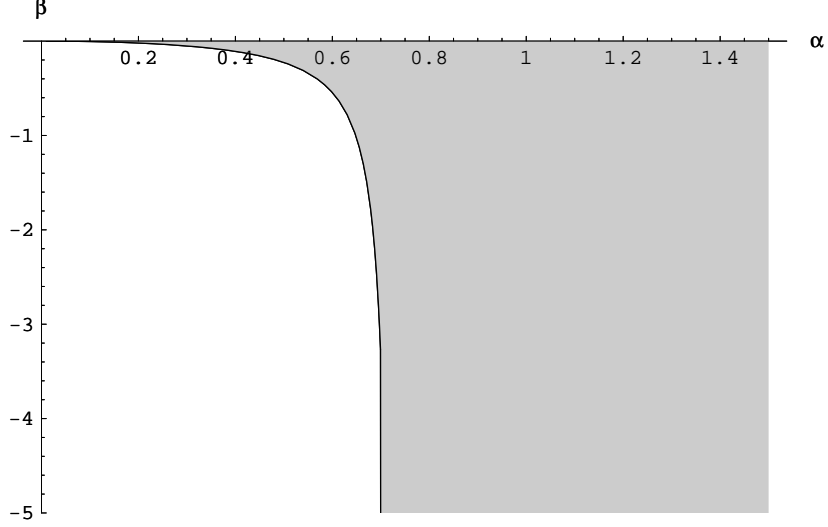
FIG. 3. The region Q_1 for the case $k = +1$, where $\rho > 0$, with $a_0 = 1$.

The pressure p can be obtained from the Eq. (8) and will not be shown here. Even $p \geq 0$ is possible if α and β are inside the region

$$Q_2 = \{\alpha > \alpha_{\min}, \beta < \beta_- \cup \beta > \beta_+\} \, , \tag{24}$$

where β_- and β_+ are functions of α and a_0, and $\alpha_{\min}$ is a function of a_0, $\alpha_{\min} \approx 0.84 \, a0$. As $\beta = 0$ is not included in the region Q_2, this result is a benefit of adding non-linear terms in the Lagrangian.

To obtain $\rho + p \geq 0$, α and β must be inside the region

$$Q_3 = \{0 \leq \alpha < \alpha_{\max}, \beta_- < \beta \leq 0\} \cup \{\alpha > \alpha_{\max}, \beta < 0\} \, , \tag{25}$$

where β_- and β_+ are another functions of α, different from the analysis of ρ, and $\alpha_{\max}$ is a function of a_0, $\alpha_{\max} \approx 0.80 \, a0$. The regions Q_1 and Q_3 are similar.

The velocity of sound does not diverge if α and β are inside the same region Q_3. But it can be imaginary for any value of $(\alpha, \beta) \neq (0,0)$; maybe only in the Q_2 region v_s has real values, this must be confirmed.

So the conclusion for the $k = +1$ case is that the addition of high-order terms in the Einstein action does not lead to any other anomaly in the effective perfect fluid, i.e., with

or without β, the behaviour of ρ, p, $\rho + p$ and v_s is the same (under the condition that α and β are inside the region Q_3).

There is even an advantage because p can be always positive, with only the problem that the region Q_2 leads to almost suppression of the radiation era. The calculus of the region Q_2 is not easy, and it is a task for further work to verify if inflationary behaviour can exist with $\rho > 0$, $p > 0$ and $0 < v_s < 1$.

III) The $k = -1$ case :

Like the $k = +1$ case, the calculations are very long, and the energy density is given by

$$\rho(a) = \frac{3\, a_0{}^2 \left(a^2 + a_0{}^2\right)\left[a^4 - 4\, a_0{}^2\, \alpha^2 + a^2 \left(a_0{}^2 - 4\sqrt{a^2 + a_0{}^2}\,\alpha\right)\right]}{\left(a^5 + a^3\, a_0{}^2 + 2\, a\, a_0{}^2\sqrt{a^2 + a_0{}^2}\,\alpha\right)^2 \kappa} + \tag{26}$$

$$\frac{216\, a_0{}^2\, \alpha\, \beta}{a^4 \left(a^4 + a^2\, a_0{}^2 + 2\, a_0{}^2\sqrt{a^2 + a_0{}^2}\,\alpha\right)^6 \kappa} \left\{ -a^6\, a_0{}^2 \left(51\, a^2 + 32\, a_0{}^2\right) \times \right.$$

$$\left(a^2 + a_0{}^2\right)^5 \alpha - 4\, a^2\, a_0{}^6 \left(a^2 + a_0{}^2\right)^4 \left(17\, a^2 + 2\, a_0{}^2\right) \alpha^3 - 16 \left(a^2 + a_0{}^2\right)^3 \times$$

$$a_0{}^{10}\, \alpha^5 + a^2 \left(a^2 + a_0{}^2\right)^{7/2} \left[a^4 \left(a^2 + a_0{}^2\right)^2 \left(5\, a^4 + 17\, a^2\, a_0{}^2 + 10\, a_0{}^4\right) - \right.$$

$$\left. \left. 8\, a^2\, a_0{}^4 \left(a^2 + a_0{}^2\right) \left(7\, a^2 + 2\, a_0{}^2\right) \alpha^2 - 48\, a_0{}^8\, \alpha^4\right]\right\} \;,$$

and $\rho \geq 0$ is impossible for $\alpha > 0$, independent of β.

The pressure p will not be shown here. Like ρ, $p \geq 0$ is impossible for $\alpha > 0$ and any β.

Obviously, for $\alpha > 0$ and any β it is not possible to obtain $\rho + p \geq 0$.

The velocity of sound is divergent and imaginary if $\alpha > 0$, independent of β.

The case $k = -1$ does not provide useful conclusions, because even without the high-order terms in the Einstein action there are a lot of anomalies in the effective perfect fluid. The hope that quadratic-order terms could avoid these anomalies was not achieved.

V. CONCLUSION

In all three cases of spatial curvature, it is possible to choose the values of α and β so that the quadratic-order terms in the Lagrangian do not add more anomalies to the effective perfect fluid.

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For closed spatial section, the opposite happens, i.e., anomalies are suppressed, and this case of non-singular inflationary scale factor with well-behaved perfect fluid deserves more investigation.

Other interesting improvements would be : use another solutions of the scale factor, more energy conditions, compare the $\beta's$ constraints with observational values and study many types of effective perfect fluids.

ACKNOWLEDGMENTS

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